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Best-Fitting Ellipsoid and Surface Disturbance Adjustment of GNSS-Derived Sea Surface Heights

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Abstract: This paper introduces a novel methodology for modeling mean sea surface topography based on GNSS-on-boat observations—continuous satellite positioning from a floating platform. The method is applied for the first time using real-world data collected in a small lagoon in Western Greece. A best-fitting ellipsoid is initially computed to represent the overall geometry of the observed sea surface. A local surface disturbance is then fitted atop the ellipsoid using a polynomial model, capturing small-scale deviations from the ideal form. The resulting hybrid surface significantly improves accuracy: initial mean discrepancies $\overline{\varepsilon}$ between the ellipsoid and GNSS measurements (3.1 cm) are reduced to below 0.1 cm after incorporating the local disturbance. This innovative approach offers a powerful tool for detailed sea surface modeling and holds promise for local geoid refinement in coastal or semi-enclosed marine areas.

Keywords: Ellipsoid, Geoid, Sea surface topography, GNSS-on-boat

Introduction

The geoid (Richards & Hager, 1984) is defined as an equipotential surface of the Earth's gravity field, which closely aligns with the global Mean Sea Level (MSL), especially in open sea regions where external influences such as winds and tides are minimal (Lycourghiotis & Stiros 2010). If MSL could be theoretically extended across landmasses via hypothetical sea canals, it would serve as a natural reference surface (Sansò et al., 2008). Accurately determining the geoid remains one of the central problems in geodesy, often referred to as the "geoid problem," due to the inherent complexity of Earth's mass distribution and limitations in observational data (Novák, 2003).

Over the years, various geometric models have been proposed to approximate the geoid, ranging from spheres to oblate spheroids and ellipsoids. Among these, the World Geodetic System 1984 (WGS84) has become the most widely adopted reference, providing a geocentric ellipsoid of revolution used as the standard framework for GNSS navigation systems and global positioning applications (Kumar, 1988; Smith, 1998).

Although WGS84 (Kelly, 2022) minimizes the global mean square deviation of Earth surface points, it inevitably produces local discrepancies in regions with complex terrain, coastlines, or seafloor variations. In such cases, the actual sea surface or geoid may deviate significantly from the WGS84 ellipsoid, leading to position errors that could be critical in high-precision applications. This has motivated the scientific community to develop high-resolution local geoid models, particularly for improving navigation, hydrographic surveys, and vertical datum realization (Galani et al., 2021; Jalal et al., 2019).

Traditionally, local geoid modeling techniques rely on known geoid undulations (N) or height differences (ΔN), often combining orthometric and ellipsoidal heights with interpolation and least squares adjustment methods.

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These approaches aim to estimate local corrections relative to the global reference surface, enhancing vertical accuracy in a given region (Lycourghiotis et al., 2024).

In this context, our study addresses the local geometric modeling of the geoid surface, focusing on semi-enclosed sea areas and coastal environments where the sea surface itself serves as a practical approximation of the geoid. In such areas, conventional gravimetric or satellite-based geoid models may exhibit reduced accuracy due to abrupt local variations in the gravity field (Hipkin, 2000).

Our approach involves computing a best-fitting triaxial ellipsoid, which shares the same center, equatorial plane, and polar axis as the WGS84 ellipsoid but is optimized to minimize geometric height deviations within the target region. This ellipsoid is estimated using the Moore–Penrose pseudoinverse, and it acts as a refined local reference surface (Dassios, 2012).

We applied this methodology in a real-world scenario using data collected from the Kotychi lagoon, a small shallow water body in Western Achaia, Greece (Figure 1). GNSS data were acquired using the GNSS-on-boat method, a high-resolution positioning system mounted on a floating platform, enabling dense and accurate sampling of the water surface over a 90-minute period. From this dataset, a best-fitting ellipsoid was calculated, and its geometric performance was assessed. To further improve the local surface representation, we introduced a polynomial surface disturbance model atop the ellipsoid, capturing subtle spatial variations not represented by the ellipsoidal fit alone. This two-step model significantly reduced the residual errors between GNSS observations and the theoretical surface, highlighting the method's potential for high-precision local geoid modeling in calm water environments.

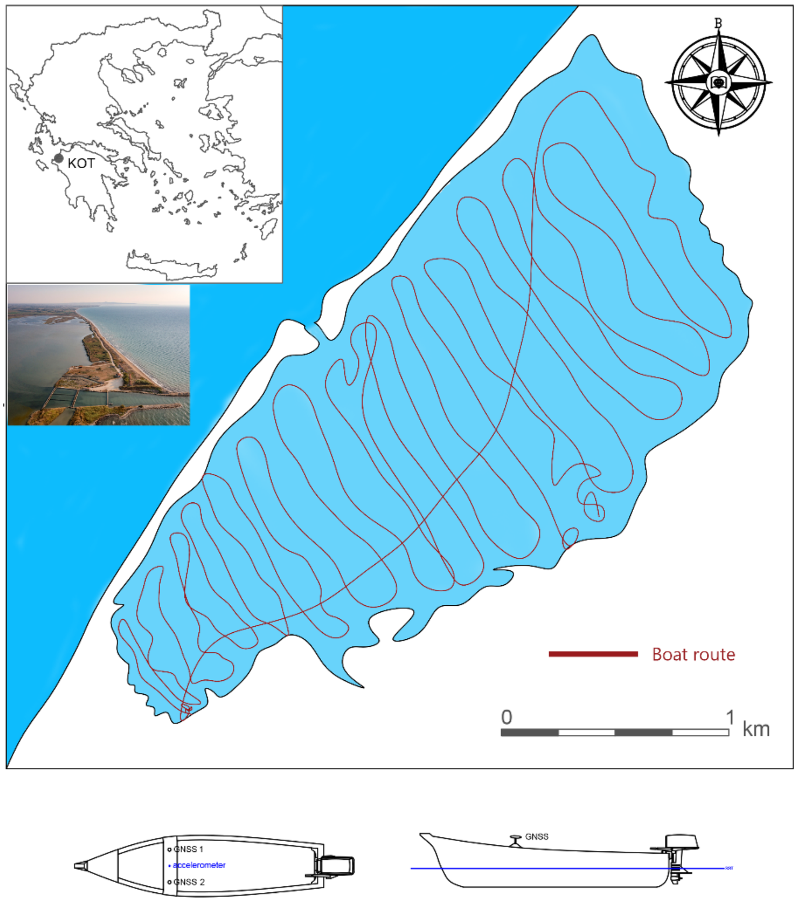

Figure 1. Location map of the experimental area. Red line represents the boat routes.

GNSS on Boat Method and Its Application in Kotychi Lagoon

The GNSS on boat (Bouin et al., 2009; Foster et al., 2009; Reinking et al., 2012) technique has emerged as a powerful tool for high precision determination of sea surface topography (SST) and near-surface geoid features, particularly in coastal and semi-enclosed water bodies (Lycourghiotis & Kariotou, 2022). The method involves mounting one or more GNSS receivers on a floating platform (boat, catamaran, wave glider etc.) and collecting high-rate kinematic positional data (typically 1 Hz or more) as the platform traverses the water surface.

Unlike traditional indirect techniques (astro geodetic deflections, tide gauges) or direct satellite altimetry approaches, GNSS on boat offers a direct measurement of the water surface height referenced to a GNSS ellipsoid, while avoiding the compounded errors inherent in two-stage systems (Chupin et al. 2020; Guo et al., 2014; Lycourghiotis, 2017a; 2017b; 2017c; Lycourghiotis & Crawford, 2024; Ocalan & Alkan, 2013). This makes the method particularly suitable for mapping SST and related geoid undulations in domains where high resolution and local detail are required, such as lagoons, coastal seas and sheltered bays (Lycourghiotis, 2021).

In the present study, we applied the GNSS on boat technique using a small 2.5 m vessel equipped with two dual frequency GNSS receivers (as depicted in the accompanying figure 1). The measurement campaign lasted approximately 90 minutes, during which the boat followed a randomized track over the Kotychi Lagoon in Western Achaia, Greece. The resulting dataset comprises a dense cloud of GNSS points capturing the mean sea surface at 1 Hz sampling rate.

In our subsequent analysis, we first derived a best-fitting triaxial ellipsoid to capture the bulk geometry of the water surface. We then enhanced the surface representation by fitting a local polynomial surface disturbance atop the ellipsoid, thereby capturing finer scale deviations between the GNSS observed surface and the theoretical ellipsoidal reference. This dual step approach allowed us to achieve a markedly improved fit and enables detailed modeling of local geoid anomalies in confined water bodies.

Methodology

To determine the best-fitted ellipsoidal in a local region, we start from determining a geocentric reference coordinate system O_{xyz} which is centered at the Earth's center of mass of Earth, polar axis zz´ the axis of rotation of the Earth and the Oxy plane coincides with the Earth's equatorial plane. We also determine a geocentric triaxial ellipsoid which has its center at the origin of the axes (0, 0, 0). If (x_i, y_i, z_i), with i = 1, 2, …, k and k are selected points in an area of our interest on Cartesian coordinates, we suppose that these points satisfy the general equation of the triaxial ellipsoid:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Replacing the components of every point in (1), we obtain a system with three unknowns, semi axis a, b, c and k equations, that is a system in form:

$$\begin{bmatrix} x_1^2 & y_1^2 & z_1^2 \\ x_2^2 & y_2^2 & z_2^2 \\ \vdots & \vdots & \vdots \\ x_k^2 & y_k^2 & z_k^2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{a^2} \\ \frac{1}{b^2} \\ \frac{1}{c^2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

The previous system is a system with more equations than unknowns therefore it has no exact solution. In this case the Moore–Penrose pseudoinverse can be used pseudoinverse matrix to find the minimum square solution (Dassios, 2012). In this way, it is found three semi axes of ellipsoidal E which approximate better the k selected points of the study region.

Having calculated the triaxial ellipsoid E, we then calculate the distance of every given point $\mathbf{r}_i = (x_i, y_i, z_i)$, $i = 1, 2, \ldots, k$, from the ellipsoid E, which is actually the geometric height h_{Ei}, of point $\mathbf{r}_i$. In particular, we calculate the minimum distance of each point $\mathbf{r}_i$ from the tangent plane of the ellipsoid E, at the point $\mathbf{r}_{di} = (x_{di}, y_{di}, z_{di})$, $i = 1, 2, \ldots, k$, on E (Dassios, 2012). In order to calculate h_{Ei}, we first find the corresponding $\mathbf{r}_{di}$.

Since the distance function between two points in three-dimensional Euclidean space is:

$$f(x_{di}, y_{di}, z_{di}) = (x_i - x_{di})^2 + (y_i - y_{di})^2 + (z_i - z_{di})^2$$

then, for $i = 1, 2, \ldots, k$, the point $\mathbf{r}_{di}$ is the one that minimizes the function f, under the condition that it belongs to the surface E, namely:

$$g(x_{di}, y_{di}, z_{di}) = \frac{x_{di}^2}{a^2} + \frac{y_{di}^2}{b^2} + \frac{z_{di}^2}{c^2} - 1 = 0$$

Using the method of Lagrange Multipliers (Brand, 2006; Marcellini, 1996) $\mathbf{r}_{di} = (x_{di}, y_{di}, z_{di})$ is given as

$$x_{di} = \frac{a^2 x_i}{a^2 - \lambda_i}, y_{di} = \frac{b^2 x_i}{b^2 - \lambda_i}, z_{di} = \frac{c^2 x_i}{c^2 - \lambda_i}.$$

The parameter λ is calculated by solving the following system

$$\begin{cases} \nabla f(x, y, z) = \lambda \cdot \nabla g(x, y, z) \\ g(x, y, z) = 0, \end{cases}$$

which yields a sixth-degree equation in λ:

$$\frac{a^2 \cdot x_i^2}{(a^2 - \lambda_i)^2} + \frac{b^2 \cdot y_i^2}{(b^2 - \lambda_i)^2} + \frac{c^2 \cdot z_i^2}{(c^2 - \lambda_i)^2} = 1$$

The real root λ corresponding to the minimum value of f is substituted into previous equation to provide the point of tangency $\mathbf{r}_{di}$, suitable for calculating the distance of point $\mathbf{r}_{di}$ from the ellipsoid E. Then, the corresponding geometric height h_{Ei}, for every point $\mathbf{r}_i$, $i = 1, 2, \ldots, k$ is given in the form:

$$h_E(x_i, y_i, z_i) = \frac{1}{\sqrt{\frac{x_{di}^2}{a^4} + \frac{y_{di}^2}{b^4} + \frac{z_{di}^2}{c^4}}} \cdot \left| \frac{x_i x_{di}}{a^2} + \frac{y_i y_{di}}{b^2} + \frac{z_i z_{di}}{c^2} - 1 \right|$$

While the best-fitting ellipsoid provides a first-order geometric approximation of the local sea surface, small-scale variations often remain unaccounted for. These may arise from local dynamic sea surface topography, atmospheric effects, tidal fluctuations, or limitations in the ellipsoidal fit. To enhance the precision of the surface model, we introduce an additional adjustment: a small vertical surface disturbance fitted atop the best-fitting ellipsoid.

Let each GNSS-observed point be $\mathbf{r}_i = (x_i, y_i, z_i)$ and let h_i^{GNSS} denote its height (e.g., ellipsoidal height from GNSS). After computing the geometric height, $h_E(x_i,y_i,z_i)$ of the point from the best-fitting ellipsoid EEE, the residual vertical disturbance is defined as:

$$\delta_i = h_i^{GNSS} - h_E(x_i, y_i, z_i)$$

These residuals δ_i represent the vertical discrepancy between the GNSS measurements and the ellipsoid-based surface and are considered to vary smoothly as a function of the horizontal coordinates.

To model the disturbance $\delta(x,y)$ we employ a bivariate polynomial approximation of degree n:

$$\delta(x, y) = \sum_{i+j \le n} a_{ij} x^i y^j$$

The model is determined by minimizing the cost function:

$$\min_{a_{ij}} \sum_{k=1}^{n} [\delta(x_k, y_k) - (h_k^{GNSS} - h_E(x_k, y_k, z_k))]^2$$

The adjusted sea surface height at each location is now modeled as:

$$h_{model}(x_i, y_i, z_i) = h_E(x_i, y_i, z_i) + \delta(x_i, y_i)$$

The revised deviation between GNSS observations and the model becomes:

$$\varepsilon_i = h_i^{GNSS} - h_{model}(x_i, y_i, z_i)$$

To test and validate the methodology described above, a dedicated algorithm was developed and implemented in the Wolfram Mathematica environment. This implementation allowed for the structured processing of GNSS data, the computation of the best-fitting ellipsoid using the Moore–Penrose pseudoinverse method, and the subsequent fitting of the local polynomial surface disturbance. The codebase, including all relevant functions and processing steps, is available upon request to any interested researchers wishing to reproduce or further develop the method.

Results

To assess the practical applicability and performance of the proposed methodology, a case study was conducted in the area of Lake Kotychi, a shallow coastal lagoon located in Western Achaia, Greece. The region was selected due to its semi-enclosed geometry, limited wave activity, and suitability for local geoid modeling. Field data were collected using the GNSS-on-boat method, where a dual-frequency GNSS receiver was mounted on a small floating platform. The receiver recorded continuous positioning data at a 1 Hz sampling rate for approximately 90 minutes, while the boat followed a randomized track across the lake. After filtering and cleaning, a dataset of 4,562 valid GNSS points was obtained, forming a dense and spatially representative cloud of height measurements over the water surface. Applying the Moore–Penrose pseudoinverse method, we computed the semi-axes of the best-fitting ellipsoid for the Kotychi lagoon as follows: $a = 6.37724 \cdot 10^6$ m, $b = 6.37929 \cdot 10^6$ m, $c = 6.35765 \cdot 10^6$ m

Following the methodology described in previous section, a best-fitting triaxial ellipsoid was computed using the Moore–Penrose pseudoinverse approach. The geometric deviations of the observed points from the ellipsoid surface were then evaluated. The resulting mean geometric height deviation $\overline{\varepsilon}$ was found to be approximately 3.1 cm. This deviation reflects the baseline discrepancy between the ellipsoidal model and the actual GNSS-observed water surface.

To improve the accuracy of the model, polynomial surface disturbances were fitted on top of the best-fitting ellipsoid, as described in previous section. Polynomial surfaces of increasing degree (from linear up to fourth-order) were tested. The residual means deviations between the GNSS data and the enhanced model (ellipsoid + disturbance) were then re-evaluated. The results are summarized below (Table 1).

Table 1. Mean deviation between GNSS observations and the model

Polynomial Degree	$\overline{\varepsilon}$
Degree 1 (linear)	0.76 cm
Degree 2 (quadratic)	0.42 cm
Degree 3 (cubic)	0.26 cm
Degree 4 (quartic)	0.08 cm

Conclusions

The best-fitting ellipsoid method has been previously shown to offer a localized geometric approximation of the sea surface based on GNSS-derived topographic data. Its strength lies in its simplicity and its ability to represent the general shape of the observed water surface with high internal consistency. However, as demonstrated in this study, the addition of a local polynomial surface disturbance significantly enhances the model's precision. By capturing small-scale deviations from the ellipsoidal shape, this hybrid approach reduces the residual fitting error to below 1 cm, and in optimal cases, to even lower sub-millimetric levels. This improvement provides the foundation for a mathematical modeling of local geoid anomalies—or more precisely, of small variations in the Earth's gravity potential field-at a resolution that is not typically accessible using traditional global or regional geoid models. Such enhanced local modeling opens new possibilities in a range of applications, including precision hydrography, coastal engineering and flood modeling, local vertical datum realization, and GNSS-based sea level monitoring in protected or semi-enclosed water bodies. Overall, the integration of a surface disturbance model over a best-fitting ellipsoid offers a robust and flexible tool for fine-resolution sea surface representation, especially in regions where dense GNSS-on-boat observations are available.

Scientific Ethics Declaration

* The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

Conflict of Interest

* The authors declare that they have no conflicts of interest

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